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EXACT AND APPROXIMATE FORMULAS FOR DEFLECTIONS OF AN ELASTICALLY FIXED ROD UNDER TRANSVERSE LOADING

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An exact solution is obtained for the nonlinear bending problem of a thin rod elastically fixed at one end and loaded by a transverse concentrated load of constant direction at the other end. The solution is written in parametric form and expressed in terms of the Jacobi elliptic functions. Based on the exact solutions, approximate formulas are proposed for the deflection of the rod tip.

Key words: *Rod bending, elastic fixing conditions, Jacobi elliptic functions.*

Introduction. The geometrically nonlinear bending problems of thin rods are treated in a general formulation in [1, 2], and the problems of rods on elastic supports and elastic foundations are considered in [3, 4]. Theory for the geometrically nonlinear bending of thin rods is developed in [5, 6]. In [6], the equilibrium configurations of a rod are determined analytically and classified for various dead loads and constraints imposed on the rod ends. The solutions obtained are written in parametric form and expressed in terms of elliptic integrals and Jacobi functions, which depend on just one parameter — the modulus of the elliptic functions, which is determined by the boundary conditions and external load, unlike in the approach of [1, 2] where the solutions depend on three parameters. The critical loads and equilibrium shapes of a rod rigidly clamped at one end and subjected to a tip follower force at the other end were obtained in [7]. The solutions mentioned above, however, are of limited use since they are inapplicable to rods with elastic constraints.

The present paper addresses the case where one end of the rod is elastically fixed and the other end is free. An exact analytical solution of the nonlinear bending problem of this rod subjected at the free end to a transverse load of constant direction. The critical loads are determined, and the equilibrium configurations of the bent rod are obtained. Approximate formulas for the deflection of the rod tip are derived.

1. Solution of the Problem of an Elastically Fixed Rod. Let us consider a thin inextensible rod of length L and flexural rigidity EI. We introduce Cartesian coordinates xy such that the initially straight rod is oriented along the x axis, is elastically fixed at the left end at the coordinate origin, and is free at the right end (Fig. 1).

The rod is bent by a transverse force P applied to the right end of the rod. The arc length of the rod from the left end to the current point will be denoted by l, and the angle between the tangent to the current point and the negative direction of the y axis by $\gamma(l)$. According to [6], the equilibrium equation of the rod is written as

$$\frac{d^2\gamma}{dt^2} + q^2 \sin\gamma = 0, \tag{1}$$

where $t = l/L$ is the dimensionless length, which varies from 0 to 1, $q^2 = PL^2/(EI)$ is the eigenvalue, and P is the magnitude of the point force.

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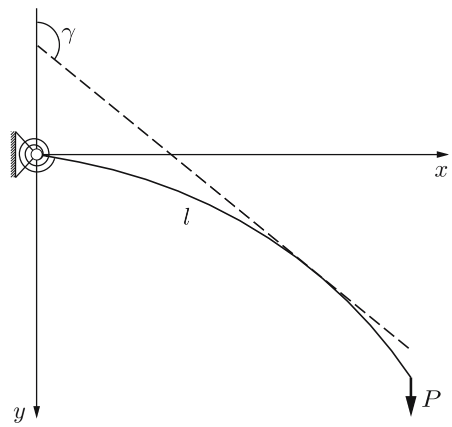

Fig. 1. Coordinate system.

At the fixed end of the rod, the boundary condition of the third kind is specified:

$$\gamma(0) - h\,\frac{d\gamma(0)}{dt} = \frac{\pi}{2}, \tag{2}$$

where $h = EI/(cL)$ is the elastic-clamping coefficient determined by the properties of the clamping fixture and the rod and c is the rotational stiffness determined by the elastic properties of the clamping fixture [3, 8]. The coefficient h thus introduced allows one to obtain the well-known boundary condition for the clamped end [6] in the case of high stiffness c ($h \to 0$). In the limiting case of small c ($h \to \infty$), one obtains the boundary condition for the hinged end.

At the free end, the bending moment vanishes:

$$\frac{d\gamma(1)}{dt} = 0. \tag{3}$$

According to [6], the solution of Eq. (1) is written as

$$\gamma = 2\arcsin\,[k\,\mathrm{sn}\,(qt + F_1)], \tag{4}$$

where sn is the Jacobi elliptic sine and the force P is expressed in terms of the modulus of the elliptic functions k and the parameter F_1 (integration constants) using the boundary conditions.

The argument of the elliptic functions will be denoted by

$$u = qt + F_1. \tag{5}$$

Using boundary condition (2), we obtain the following transcendental equation for the integration constant F_1:

$$2\arcsin\,(k\,\mathrm{sn}\,F_1) - 2hkq\,\mathrm{cn}\,F_1 = \pi/2. \tag{6}$$

Condition (3) implies that $\mathrm{cn}\,(q + F_1) = 0$, which yields

$$q = (2n-1)K(k) - F_1 \tag{7}$$

[$n = 1, 2, 3, \ldots$ is the solution mode number and $K(k)$ is the complete elliptic integral of the first kind]. Solution of system (6), (7) gives the eigenvalue spectrum $q_n(k)$, which in turn determines the expression for the normalized loads corresponding to the nth mode:

$$P/P_{\mathrm{cr}} \equiv (2/\pi)^2 q^2 = (2/\pi)^2[(2n-1)K(k) - F_1]^2. \tag{8}$$

Here $P_c = (\pi/2)^2 EI/L^2$ is the Euler critical load. Expression (8) has the same structure as the expression for the normalized loads for a cantilever rod under loading of constant direction [6]. For transverse loading, the parameter k varies in the range $1/2 \leqslant k^2 \leqslant 1$. The parameter F_1 is determined numerically by solving system (6), (8) for given values of the elliptic modulus k and the coefficient h. For various ranges of the load P/P_{cr} between successive

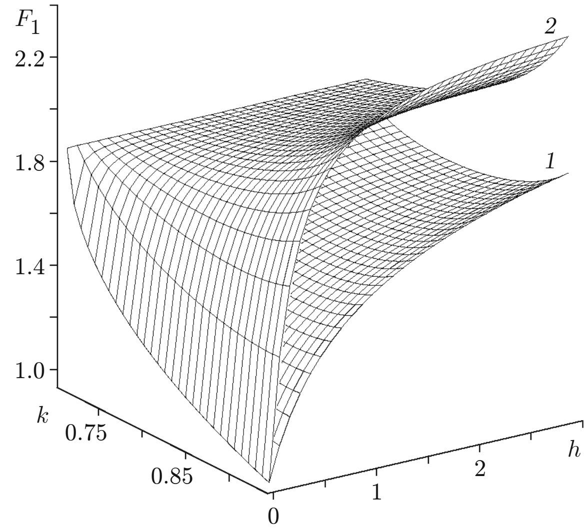

Fig. 2. The integration constant F_1 as a function of the elliptic modulus k and the elastic-fixing coefficient h for the first and second modes: curves 1 and 2 refer to $n = 1$ and 2, respectively.

thresholds, which can be determined using a linear approximation, we obtain surfaces "glued" along two lines: 1) for the threshold value of the modulus $k^2 = 1/2$; 2) for $h = 0$. Figure 2 shows these surfaces for modes $n = 1$, 2.

For the case of rigid clamping represented by the curve $h = 0$ in Fig. 2, the parameter F_1 is given by [6] $F_1 = F[\arcsin(\sqrt{2}/2k), k]$. For $k^2 = 1/2$, which corresponds to the threshold loads, and any value of the elastic-clamping coefficient h (curve $k = \sqrt{2}/2$ in Fig. 2), we have $F_1 = K(\sqrt{2}/2)$. In this case, relation (8) leads to the following equation for the threshold loads corresponding to the nth mode [6]:

$$P_n/P_{\rm cr} = (n-1)^2[(4/\pi)K(\sqrt{2}/2)]^2 \approx 5.6(n-1)^2. \tag{9}$$

Figure 3 shows the normalized load (8) versus the elliptic modulus k for various values of h for the first and second modes. As the coefficient h increases, the surfaces corresponding to the first and second modes are located lower than in the case $h = 0$. This implies that for the elastic fixing conditions, deflections of the same magnitude occur for lower loads than in the clamped case.

Integration of the relations $dx/dl = \sin\gamma$ and $dy/dl = \cos\gamma$ with the use of Eq. (5) yields the coordinates of an arbitrary point of the rod

$$\begin{aligned} \frac{x}{L} &= \frac{2k}{(2n-1)K(k) - F_1}\,(\operatorname{cn} F_1 - \operatorname{cn} u), \\ \frac{y}{L} &= t - \frac{2}{(2n-1)K(k) - F_1}\,(E(\operatorname{am} u, k) - E_1(k)), \end{aligned} \tag{10}$$

where $E(\operatorname{am} u, k)$ and $E_1(k) = E(\operatorname{am} F_1, k)$ are incomplete elliptic integrals of the second kind of the elliptic amplitude and t is the normalized length of the rod. Expressions (10) define the bent rod configurations in parametric form.

Figure 4 shows equilibrium shapes of the elastically fixed rod corresponding to various values of the applied force for $h = 0.4$ for the first and second modes.

1.1. *Coordinates of the Inflection Points.* The points at which the second derivative d^2y/dx^2 vanishes are the points of inflection of the rod axis. We denote the curvilinear coordinate of such a point by t_1. Using (4), we evaluate the second derivative of the function describing the bent axis of the rod determined by the parametric equations (9) of the form $x = x(t)$ and $y = y(t)$:

$$\frac{d^2y}{dx^2} = \frac{x'y'' - y'x''}{x'^3} = \frac{1}{\sin^3\gamma}\,\frac{d\gamma}{dt} = \frac{2kq\operatorname{cn} u}{\sin^3\gamma}.$$

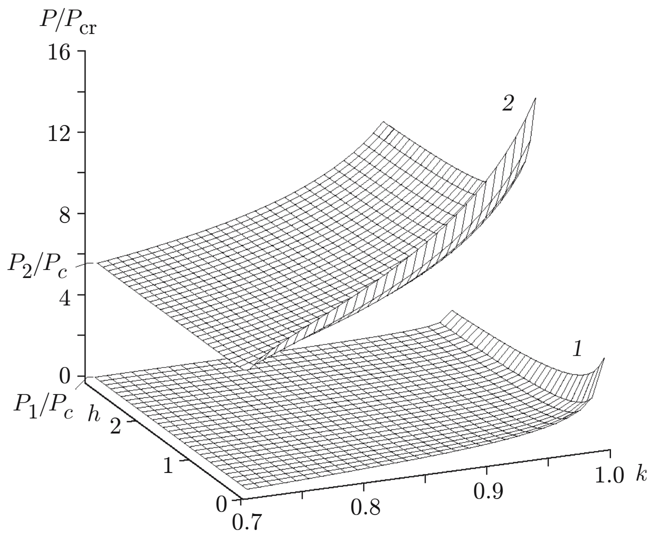

Fig. 3

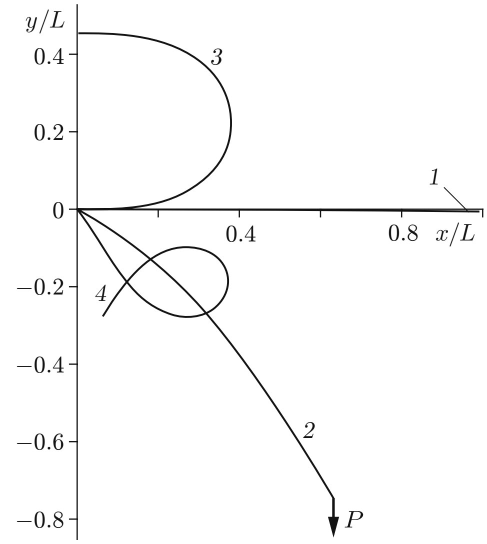

Fig. 4

Fig. 3. Eigenvalue spectrum for the equilibrium equation of an elastically fixed rod under transverse loading for the first and second modes: curves 1 and 2 refer to $n = 1$ and 2, respectively.

Fig. 4. Shapes of a cantilever for $h = 0.4$ and various loads $P/P_{\rm cr}$: curves 1 and 2 refer to the first mode (curve 1 refers to $P/P_{\rm cr} = 0.004$ and curve 2 to $P/P_c = 0.79$) and curves 3 and 4 refer to the second mode (curve 3 refers to $P/P_c = 5.58$ and curve 4 to $P/P_c = 15.48$).

Taking into account the properties of the zeroes of the Jacobi elliptic cosine $\operatorname{cn}[(2m+1)K(k)] = 0$ and using expression (7) for the eigenvalue q, from the last expression we obtain

$$[(2n-1)K(k) - F_1]t_1 + F_1 = (2m+1)K(k),$$

where $n = 1, 2, 3, \ldots$ and m enumerates zeroes of the elliptic cosine and is equal to the ordinal number of the inflection point. As a result, we have

$$t_1^{nm} = \frac{(2m+1)K(k) - F_1}{(2n-1)K(k) - F_1}$$

[F_1 is determined from Eq. (6)].

The number of inflection points for a mode is equal to the mode number. The inflection point number m depends on the mode number and varies in the range from 0 to $n-1$. For the first mode, one inflection point occurs at the rod tip: $t_1^{10} = 1$, for the second mode, a second inflection point occurs, whose location on the rod axis changes for $1/2 \leqslant k^2 \leqslant 1$, and for the third mode, there are three inflection points.

For a rigidly clamped rod, the coordinates of moving inflection points are found in [5] to be in the intervals $0 \leqslant t_1^{20} \leqslant 1/3$, $t_1^{21} = 1$ and $t_1^{21} = 1$ for $n = 2$ and in the intervals $0 \leqslant t_1^{30} \leqslant 1/5$, $1/2 \leqslant t_1^{31} \leqslant 3/5$, and $t_1^{32} = 1$ for $n = 3$. These intervals correspond to the case of an elastically fixed cantilever for $h = 0$.

1.2. *Coordinates of the Compression Points.* By compression points we understand the points at which the slope of the tangent to the x axis is equal to that of the compressive force. We denote the unknown curvilinear coordinate of this point by t_0. Setting $\gamma(t_0) = 0$ in the expression for the tangent slope (4), we obtain

$$0 = 2 \arcsin [k \operatorname{sn}(qt_0 + F_1)].$$

Using the properties of the zeroes of the elliptic sine $\operatorname{sn}[2mK(k)] = 0$ and expression (7) for the eigenvalue q, we obtain

$$[(2n-1)K(k) - F_1]t_0 + F_1 = 2mK(k).$$

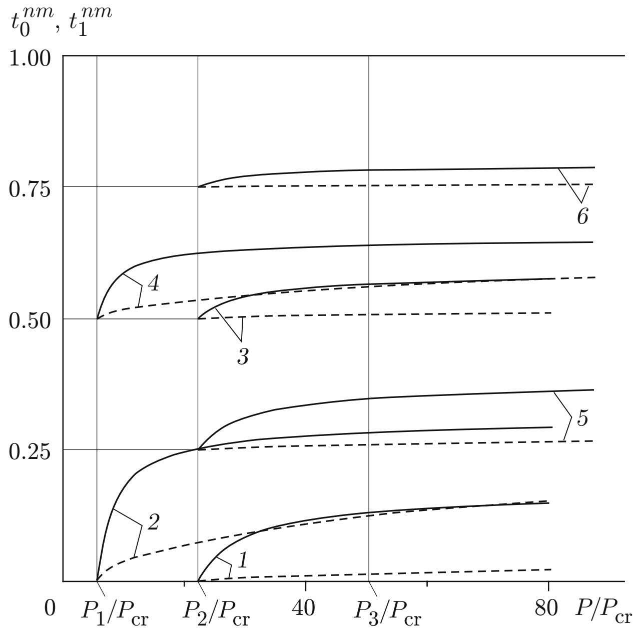

Fig. 5

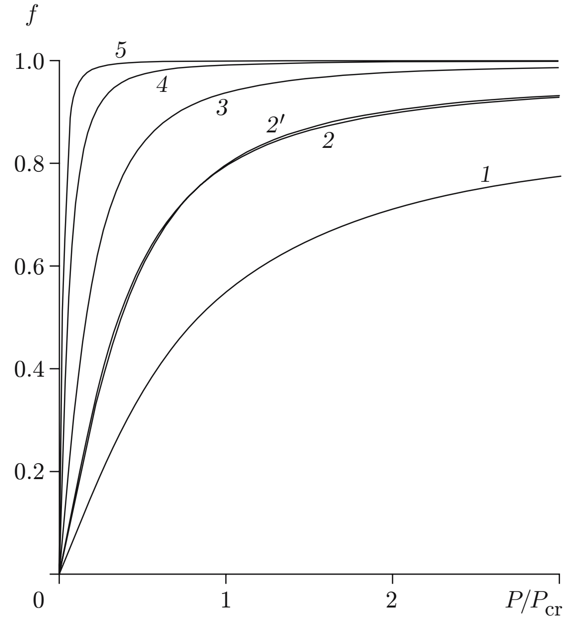

Fig. 6

Fig. 5. Coordinates of the inflection points t_1^{nm} and compression points t_0^{nm} versus transverse load for various values of h: solid curves refer to $h = 0$ and dashed curves refer to $h = 1.2$; curve 1 refers to t_1^{30}, curve 2 to t_1^{20}, curve 3 to t_1^{31}, curve 4 to t_0^{21}, curve 5 to t_0^{31}, and curve 6 to t_0^{32}.

Fig. 6. Tip deflection versus P/P_c for various values of h: curve 1 refers to $h = 0$ and curves 2 and $2'$ refer to $h = 0.2$ [curve $2'$ is obtained by approximate formula (31)], curve 3 refers to $h = 1.2$, curve 4 to $h = 4.5$, and curve 5 to $h = 7$.

Here $n = 1, 2, 3, \ldots$ and m enumerates the elliptic-sine zeroes and is equal to the ordinal number of the compression point. As a result, we have

$$t_0^{nm} = \frac{2mK(k) - F_1}{(2n-1)K(k) - F_1}, \tag{11}$$

[F_1 is determined from Eq. (6)].

For a mode, the number of compression points is equal to $n - 1$. The compression-point number takes the values $m = 1, \ldots, n-1$. There are no compression points for the first mode. For $n = 2$, one compression point occurs in the interval $1/2 \leqslant t_0^{21} \leqslant 2/3$ for $1/2 \leqslant k^2 \leqslant 1$; for $n = 3$, two compression points occur in the intervals $1/4 \leqslant t_0^{31} \leqslant 2/5$ and $3/4 \leqslant t_0^{32} \leqslant 4/5$.

Figure 5 shows the coordinates of the inflection and compression points versus the load $P/P_{\rm cr}$. The vertical lines in Fig. 5 correspond to the threshold values of the loads (8). The horizontal lines correspond to the lower bounds of the intervals of compression and inflection points. (For all these lines, we have $k^2 = 1/2$.) As h increases, the upper bounds of these intervals decrease and coincide with the lower bound as $h \to \infty$. In other words, as $h \to \infty$, the coordinates of the inflection and compression points do not depend on the magnitude of the applied force and take the values $t_1^{nm} = m/(n-1)$ and $t_0^{nm} = m/(2(n-1))$, respectively, which correspond to the location of the points on the rod axis for the threshold loads $P_n/P_{\rm cr}$. For any values of the coefficient h, the load P/P_c, and the mode number n, an inflection point occurs at the free tip of the cantilever.

1.3. *Tip Deflection.* The exact expression for the deflection of the rod tip is given by

$$f(k) \equiv \frac{y(1)}{L} = 1 - 2\,\frac{E(k) - E_1(k)}{K(k) - F_1}, \tag{12}$$

where $E(k)$ is the complete elliptic integral of the second kind.

Figure 6 shows tip deflection versus load for various values of the coefficient h. One can see that for the same loads, the tip deflection of the elastically fixed rod increases with increasing h.

2. Approximate Expressions for the Tip Deflection. We obtain approximate formulas for the elliptic integrals appearing in the expression for the tip deflection (12) for two limiting values of the coefficient h. We consider expression (6), which determines the parameter F_1 for the first mode.

2.1. *Case of Small Values of the Coefficient* h $(h \to 0)$. Setting $h = 0$ in (6), we obtain $F_1 = F_0$ and $F_0 = F(\arcsin[\sqrt{2}/(2k)])$. For a small positive change in the coefficient h in the vicinity of zero, we write F_1 as $F_1 = F_0 \pm \Delta F$, where ΔF is a small quantity. Expansion of the terms in (6) in a Taylor series in terms of the small parameter ΔF yields

$$\arcsin[k \operatorname{sn}(F_0 \pm \Delta F)] \approx \pi/4 \pm (4k^2 - 2)^{1/2} \Delta F/2,$$

$$\operatorname{cn}(F_0 \pm \Delta F) \approx (4k^2 - 2)^{1/2}/(2k) \mp (1/(2k)) \Delta F.$$

Substitution of these expansions into (6) taking into account that $\Delta F \to 0$ as $h \to 0$ leads to the approximate expression

$$F_1 \approx F_0 + (K(k) - F_0)h. \tag{13}$$

Since $E_1 = E(\operatorname{am} F_1, k)$, expansion in terms of the small parameter ΔF yields

$$E_1 \approx E_0 + (K(k) - F_0)h/2, \tag{14}$$

where $E_0 = E(\arcsin[\sqrt{2}/(2k)])$. Using expressions (12)–(14), we obtain the following approximate expression for the tip deflection of the elastically fixed rod subjected to transverse loading for small values of the coefficient h:

$$f(k) \approx 1 - 2\,\frac{(E(k) - E_0) - (K(k) - F_0)h/2}{(K(k) - F_0)(1 - h)}. \tag{15}$$

For $h \to 0$, relation (15) expresses the tip deflection of the rigidly clamped rod under transverse loading [9].

Using the expansions of the differences of the elliptic integrals $E(k) - E_0$ and $K(k) - F_0$ in series in the small parameter $\xi = (k - \sqrt{2}/2)$ [9] and keeping the first three terms in these expansions, we obtain the approximate formula for the tip deflection

$$f(k) \approx 1 - \frac{240 - 260\sqrt{2}\xi + 17\xi^2}{(240 + 60\sqrt{2}\,\xi + 497\xi^2 + \ldots)(1 - h)} + \frac{h}{1 - h}. \tag{16}$$

In a similar manner, we linearize expression (8) for the normalized load P/P_{cr}. Using the approximate expression (13) for F_1 and introducing the notation for the normalized load $\lambda \equiv P/P_{\mathrm{cr}}$, we have

$$\lambda = (2/\pi)^2 (1 - h)^2 [K(k) - F_0]^2.$$

Using the results of [9] for the expansion of $K(k) - F_0$ in a series in the small parameter ξ up to and including the second term, we arrive at the cubic equation

$$(\sqrt{\xi}\,)^3 + 2^{3/2}\sqrt{\xi} - \pi\sqrt{\lambda}/(2^{3/4}(1 - h)) = 0. \tag{17}$$

Bearing in mind the condition $1/2 \leqslant k^2 \leqslant 1$, we solve Eq. (17) for $\sqrt{\xi}$ by the Ferro–Cardano formula and expand it in a series in the small parameter λ, we obtain the approximate relation between the parameter ξ and the normalized load λ:

$$\xi \approx \Big(\frac{\pi}{2}\Big)^2 \frac{\sqrt{2}}{(1 - h)^2} \Big(\frac{\lambda}{8} - \frac{\lambda^2}{64(1 - h)^2} + \ldots\Big). \tag{18}$$

Substituting (18) into (16), expanding the result in a series in the small parameter λ, and retaining the first two terms in the series, we arrive at the following approximate relation between the tip deflection and the normalized load:

$$f(\lambda) \approx \frac{\pi^2}{12(1 - h)^3}\,\lambda + \frac{\pi^2(\pi^2 - 4)}{384(1 - h)^5}\,\lambda^2 + \ldots\ . \tag{19}$$

In the case of a rigidly clamped rod, the linear term in (19) becomes an approximate expression for rod deflection under transverse loading, which is well-known in the theory of material strength. For small values of the coefficient h, formula (19) can be used to determine the deflection whose magnitude does not exceed 20% of the rod length.

2.2. *Case of Large Values of the Coefficient* h $(h \to \infty)$. Letting $h \to \infty$ in (6), we obtain $F_1 = K(k)$. Hence, for infinite positive values of the coefficient h, the quantity F_1 can be written as $F_1 = K(k) \pm \Delta F$, where ΔF is a small term. Expansion of the terms in (6) in a Taylor series in the small parameter ΔF yields

$$\arcsin\left[k\,\mathrm{sn}\,(K(k) \pm \Delta F)\right] \approx \arcsin k - (1/2)k(1-k^2)^{1/2}\Delta F,$$

$$\mathrm{cn}\,(K(k) \pm \Delta F) \approx \mp(1-k^2)^{1/2}\Delta F.$$

Substitution of these expansions into (6) yields

$$\Delta F = \pm R/h^{1/2},$$

where $R = \sqrt{(\arcsin k - \pi/4)/(k\sqrt{1-k^2}\,)}$.

Since $F_1 = K(k) \pm \Delta F$ and $E_1 = E(\mathrm{am}\,F_1, k)$, we obtain the approximate expressions

$$F_1 = K(k) \pm R/h^{1/2}, \qquad E_1 = E(k) \pm (1-k^2)R/h^{1/2}. \tag{20}$$

In view of (12) and (20), the approximate expression for the tip deflection of the elastically fixed rod under transverse loading becomes

$$f(k) \approx 1 - 2(1-k^2). \tag{21}$$

Expansion of expression (21) in a series in the small parameter ξ yields

$$f(k) \approx 2\sqrt{2}\,\xi + 2\xi^2 + \dots\ . \tag{22}$$

From (22) it follows that as $k^2 \to 1/2$ $(P \to 0)$, the parameter ξ and the tip deflection tend to zero and as $k^2 \to 1$ $(P \to \infty)$, the tip deflection is equal to unity according to (21).

Substitution of (20) into (8) yields

$$\lambda \approx 4R^2/(\pi^2 h). \tag{23}$$

Expanding in a series with respect to the small parameter ξ up to the second order, we write expression (23) as the quadratic equation

$$\xi^2 + \sqrt{2}\,\xi - \pi^2\lambda h/8 = 0. \tag{24}$$

Solution of Eq. (24) for ξ taking into account that $1/2 \leqslant k^2 \leqslant 1$ gives the following expression of ξ in terms of the normalized load:

$$\xi \approx (\sqrt{2}/2)(\sqrt{1+\pi^2\lambda h/4} - 1). \tag{25}$$

Substitution of (25) into (22) yields the relation between the tip deflection and the load for large values of the coefficient h and small values of the load λ. In this case, this linearization technique allows one to obtain only the linear term in the series expansion in λ:

$$f(\lambda) = (\pi^2 h/4)\lambda + \dots\ . \tag{26}$$

2.3. *Case of Small Loads* λ $(\lambda \to 0)$. The case $\lambda \to 0$ is equivalent to the limiting transitions as $k^2 \to 1/2$ or $\xi \to 0$. We linearize the exact expression (6) with allowance for (7) by expanding in a series in the small parameter λ. To this end, we replace q and F_1 in (6) using the relations $\lambda = (2/\pi)^2 q^2$ and $F_1 = K(k) - (\pi/2)\sqrt{\lambda}$ at $n = 1$. As a result, we have

$$2\arcsin\left[k\,\mathrm{sn}\,(K - (\pi/2)\sqrt{\lambda})\right] - 2hk(\pi/2)\sqrt{\lambda}\ \mathrm{cn}(K - (\pi/2)\sqrt{\lambda}\,) = \pi/2. \tag{27}$$

Expanding expression (27) in a series in the small parameter λ and restricting our attention to the linear term in the series, we obtain the following approximate formula for the normalized load:

$$\lambda \approx \frac{8(k - \sqrt{2}/2)}{\pi^2 k\sqrt{1-k^2}\,(\sqrt{2}h + \sqrt{1-k^2})}. \tag{28}$$

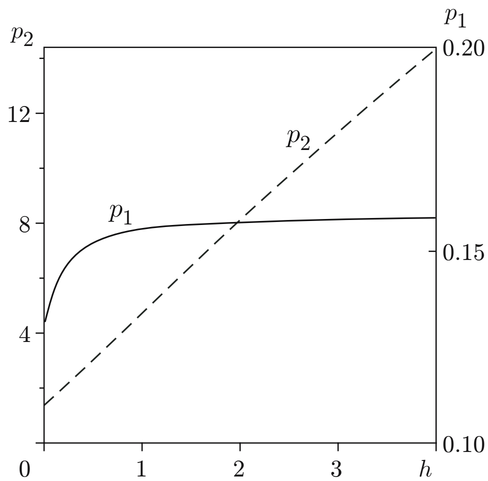

Fig. 7. Coefficients p_1 and p_2 versus h.

Expanding (28) in a series in the neighborhood of ξ up to the second order of smallness, we write the resulting expression as a quadratic equation for ξ:

$$\xi^2 + \frac{2h+1}{\sqrt{2}}\,\xi - \frac{\pi^2(2h+1)^2\lambda}{32} = 0. \tag{29}$$

Solving Eq. (29) for ξ and bearing in mind that $1/2 \leqslant k^2 \leqslant 1$, we obtain the following relation between ξ and the normalized load:

$$\xi \approx \sqrt{2}(2h+1)(\sqrt{1+\pi^2\lambda/4} - 1)/4. \tag{30}$$

Using (7) and (12), we express the tip deflection in terms of the modulus k and the loading parameter λ:

$$f(\lambda) = 1 - 4(E(k) - E(\operatorname{am}[K(k) - (\pi/2)\sqrt{\lambda}]))/(\pi\sqrt{\lambda}\,). \tag{31}$$

Expanding expression (31) in a series in the parameters ξ and λ and neglecting terms higher than the second order of smallness, we obtain

$$f(\lambda) \approx -\frac{\pi^2\lambda}{24} + \Big(-\frac{\pi^4\sqrt{2}\lambda^2}{240} + 2\sqrt{2}\Big)\xi + \Big(\frac{\pi^2\lambda}{3} + 2 - \frac{\pi^4\lambda^2}{240}\Big)\xi^2 + \ldots\ . \tag{32}$$

Using (30) and (32), we obtain the following approximate expression for the deflection for small loads λ and any h:

$$f(\lambda) \approx \frac{\pi^2(3h+1)}{12}\,\lambda + \frac{\pi^4(64h^2-1)}{256}\,\lambda^2 + \ldots\ . \tag{33}$$

Expression (33) is a generalization of the approximate results (19) and (26) for the case of arbitrary h. Compared to (33), relation (19) provides higher accuracy in determining the tip deflection for small values of h.

Using the asymptotic approximations obtained in [9], we construct the following approximate relation between the tip deflection and the load, in which the coefficients are determined by the nonlinear regression method for particular values of h:

$$f(\lambda) = p_1(\exp(2(1 - 1/(p_2\lambda+1)^2)) - 1). \tag{34}$$

The dependence of the coefficients p_1 and p_2 on h in (34) is shown in Fig. 7 (the root-mean-square error does not exceed 0.0004 for each case). The error in determining the tip deflection by formula (34) for a given load (which can exceed three critical loads) is smaller than 3%.

Conclusions. Exact analytical expressions governing the shapes of an elastically fixed flexible rod bent by a transverse dead load were obtained using the approach of [6]. In contrast to the results of [5, 6], the solution for the deflections of an elastically fixed rod depends not only on the elliptic modulus k, which is determined by the external load, loading direction, and solution mode, but it also depends on the elastic-clamping coefficient h. Approximate formulas were derived for the tip deflection of an elastically fixed rod for limiting cases.

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REFERENCES

1. E. P. Popov, *Theory and Calculation of Flexible Elastic Rods* [in Russian], Nauka, Moscow (1986).
2. E. P. Popov, *Nonlinear Static Problems of Thin Rods* [in Russian], OGIZ, Leningrad–Moscow (1948).
3. W. Weaver, S. P. Timoshenko, D. H. Young, *Vibration Problems in Engineering*, John Wiley and Sons (1990).
4. A. S. Vol'mir, *Stability of Deformable Systems* [in Russian], Nauka, Moscow (1967).
5. Yu. V. Zakharov and A. A. Zakharenko, "Dynamic instability in the nonlinear problem of a cantilever," *Vychisl. Tekhnol.*, **4**, No. 1, 48–54 (1999).
6. Yu. V. Zakharov and K. G. Okhotkin, "Nonlinear bending of thin elastic rods," *J. Appl. Mech. Tech. Phys.*, **43**, No. 5, 739–744 (2002).
7. Yu. V. Zakharov, K. G. Okhotkin, and A. D. Skorobogatov, "Bending of bars under a follower load," *J. Appl. Mech. Tech. Phys.*, **45**, No. 5, 756–763 (2004).
8. A. P. Seiranyan, "Lagrange problem of the most rational shape of a column," *Usp. Mekh.*, No. 2, 45–96 (2003).
9. Yu. V. Zakharov, K. G. Okhotkin, and A. Yu. Vlasov, "Approximate formulas for sagging deflections of an elastic rod under transverse loading," *J. Appl. Mech. Tech. Phys.*, **43**, No. 5, 745–747 (2002).